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Dispersion equations and a comparison of different quasi-periodic solutions of the sine-Gordon equation

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Abstract. A system of dispersion equations for quasi-periodic solutions of the multidimensional sine-Gordon equation is discussed. This system of algebraical equations determines the parameters appearing in the solution which involves abstract theta-functions. In the case of the two-phase quasi-periodic solutions, it is shown that the form involving theta-functions represents a broader class than the class of solutions given by the expression $4\tan^{-1} fg$. The condition for the equivalence of both classes is also reported.

1. Introduction

This paper concludes an idea developed in previous papers (Zagrodziński 1981, 1982, to be referred to hereafter as I and II). We discuss the system of algebraical equations which is equivalent to the multidimensional sine-Gordon (SG) equation, if one looks for quasi-periodic solutions in the form

$$\Psi = 2\mathrm{i}\ln[\theta(\boldsymbol{z}+\tfrac{1}{2}\boldsymbol{d}|B)/\theta(\boldsymbol{z}|B)] + (1\mp 1)\pi/2, \tag{1}$$

where $\theta(\boldsymbol{z}|B)$ is the multidimensional theta-function. The vector $\boldsymbol{z}$ is a g-dimensional vector

$$z_i = \sum_{p=1}^{N} a_{ip}x_p + z_{i0}, \qquad i = 1, 2, \ldots, g, \tag{2}$$

and we identify the last component of the N-dimensional vector $\boldsymbol{x}$ as time, i.e. $x_N = \mathrm{i}t$. $\boldsymbol{d}$ represents the unit vector

$$\boldsymbol{d} = (1, 1, \ldots, 1), \tag{3}$$

and B is a Riemann matrix (see appendix 1); $\boldsymbol{z}_0$ is an arbitrary constant vector.

Since quasi-periodic solutions have an analogy with soliton ones and the SG equation is important in many branches of physics, the aim of this paper is to determine the solution (1) completely, i.e. to determine also the constants a_{ip} appearing in the expression (1) via (2).

It has already been shown in I and II that upon substitution of (1) into the multidimensional SG equation one obtains a system of algebraic equations which involve the constants a_{ip} indirectly.

It turns out that the system obtained is linear with respect to the quantities

$$A_{pq} = \sum_{i=1}^{g} a_{ip}a_{iq}, \tag{4}$$

which one can interpret as the scalar products of the vectors a_p and a_q. Since for $N=2$, $z_i = a_{i1}x - a_{i2}t + z_{i0}$, $1 \leq i \leq g$, and a_{i1} has the character of a propagation vector, whereas a_{i2} has that of an angular frequency, for the g-periodic wave process, the discussed system of algebraical equations plays the role of a system of dispersion equations for the SG equation.

Solution (1) for the (1+1)-dimensional SG equation was derived by Kozel and Kotlyarov (1976) and then quoted by many authors (Matveev 1976, Nakamura 1979, 1980, Zakharov *et al* 1980), particularly as an example of an application of abelian functions to the theory of nonlinear partial differential equations. The generalisation for the case of the multidimensional SG equation, based on a completely different approach, was considered in I and II. Quite recently the properties of (1+1)-dimensional solutions were related to the structure of the spectrum of an associated scattering problem (Forest and McLaughlin 1980).

To our knowledge the system of dispersion equations for the SG equation has been reported only in I and II.

In § 2 we derive an equivalent form of dispersion equations which we believe to be more convenient for further analysis and more effective for numerical evaluations. In § 3, we compare the different forms of quasi-periodic solutions of the SG equation.

As was pointed out in I, if there exists a correlation between the elements of the matrix B, or if the off-diagonal elements have a particular form, the multi-dimensional θ-function may be 'split' into a finite sum of the products of the fewer-dimensional θ-functions. This fact is fundamental for a comparison in the simplest 1+1 case of the solution given by (1) and the commonly known, 'old' solution given by $\Psi = 4\tan^{-1} f(x)g(t)$.

It turns out that the two-phase quasi-periodic solutions of type (1) form a broader class, since only in particular cases can the two-periodical θ-function be represented by one-dimensional Riemann ϑ-functions. On the other hand the 'old' solutions, as expressed by elliptic functions, have always a representation in terms of the one-dimensional ϑ-functions. Both forms of solutions coincide if the diagonal elements of the B matrix are equal and an elucidation of these questions concludes § 3.

Furthermore in the appendices we summarise the essential relations for the θ-functions which can be helpful for the reader.

2. An algebraic representation of the SG equation

For the sake of completeness, a concise review of the main topics covered in papers I and II is in order. There we considered the multidimensional SG equation

$$\sum_{i=1}^{N} \partial_{x_i}^2 \Psi = \sin \Psi, \tag{5}$$

where $x_N = \mathrm{i}t$. $x_1, \ldots, x_{N-1}$ are interpreted as the space coordinates and t as time. Thus equation (5) concerns $((N-1)+1)$-dimensional space–time.

Upon introduction of the new variables z_i related formally to x_p by (2), the SG equation becomes

$$\sum_{p,q=1}^{g} A_{pq}\partial_{z_p}\partial_{z_q}\Psi = \sin \Psi, \tag{6}$$

and the coefficients A_{pq} are given just by (4). Observe that g, the fixed number of the independent z_i variables, may be different from N and is a parameter in further considerations. Thus, to be precise, we look for the g-phase quasi-periodic solution of the SG equation in the $(N-1+1)$-dimensional space–time.

Instead of the term 'g-phase quasi-periodic solution', the terms 'solutions on a circle', 'kink-trains', 'wavetrains', or simply 'periodic solutions' are also in commonly accepted usage.

The next step was the most important. By the substitution of Ψ given by (1) in the relation (6) one can determine A_{pq} and further, by (4), also a_{ip}. Indeed, making use of (A1.9), equation (6) reduces to

$$\sum_{\boldsymbol{\varepsilon}\in \boldsymbol{D}^g}\left(\sum_{p,q=1}^{g} A_{pq}\Omega_{\boldsymbol{\varepsilon}}^{pq}\pm\tfrac{1}{4}\delta_{\boldsymbol{\varepsilon},\boldsymbol{d}}\right)F(\boldsymbol{z};\boldsymbol{\varepsilon}|B)=0, \tag{7}$$

where the outer sum over $\boldsymbol{\varepsilon}\in\boldsymbol{D}^g$ here and later denotes the sum over all g-dimensional vectors $\boldsymbol{\varepsilon}$ with components ε_i taking only the two values 0 or 1. $\delta_{\boldsymbol{\varepsilon},\boldsymbol{d}}$ is the Kronecker symbol and the functions $F(\boldsymbol{z};\boldsymbol{\varepsilon}|B)$ are given by the squares of multidimensional θ-functions

$$F(\boldsymbol{z};\boldsymbol{\varepsilon}|B)=\theta^2(\boldsymbol{z}+\tfrac{1}{2}\boldsymbol{d}+\tfrac{1}{2}\boldsymbol{\varepsilon}|B)\theta^2(\boldsymbol{z}|B)-\theta^2(\boldsymbol{z}+\tfrac{1}{2}\boldsymbol{d}|B)\theta^2(\boldsymbol{z}+\tfrac{1}{2}\boldsymbol{\varepsilon}|B). \tag{8}$$

The coefficients $\Omega_{\boldsymbol{\varepsilon}}^{pq}$ come from the differentiation of a θ-function (see A1.9) and (A1.11)), as

$$\Omega_{\boldsymbol{\varepsilon}}^{pq}=2^{-(g+1)}\sum_{\boldsymbol{\delta}\in\boldsymbol{D}^g}(-1)^{\boldsymbol{\varepsilon}\cdot\boldsymbol{\delta}}\theta^{-1}(B\boldsymbol{\delta}|2B)\partial_{w_p}\partial_{w_q}[\theta(2\boldsymbol{w}+B\boldsymbol{\delta}|2B)\exp(\mathrm{i}2\pi\boldsymbol{w}\cdot\boldsymbol{\delta})]|_{\boldsymbol{w}=0}, \tag{9}$$

where the symbol $\boldsymbol{\varepsilon}\cdot\boldsymbol{\delta}$ denotes the scalar product, e.g.

$$\boldsymbol{\varepsilon}\cdot\boldsymbol{\delta}=\sum_{i=1}^{g}\varepsilon_i\delta_i, \tag{10}$$

and ∂_{w_i} the differentiation with respect to the ith component of the vector $\boldsymbol{w}$.

Observe now that (7) is satisfied if for each non-vanishing $\boldsymbol{\varepsilon}\in\boldsymbol{D}^g$

$$\sum_{p,q=1}^{g}A_{pq}\Omega_{\boldsymbol{\varepsilon}}^{pq}=\mp\tfrac{1}{4}\delta_{\boldsymbol{\varepsilon},\boldsymbol{d}}. \tag{11}$$

This equation would be a unique condition for the fulfilment of (7) if $F(\boldsymbol{z};\boldsymbol{\varepsilon}|B)$ were linearly independent. Since for $\boldsymbol{\varepsilon}=0$, by (8) it follows that $F(\boldsymbol{z};\boldsymbol{0}|B)=0$, formula (11) represents a system of 2^g-1 algebraic equations which apparently can be treated as a system of dispersion equations for the SG equation.

In II it was shown that for $g=1$ or 2, equations (11) have always a solution for each Riemann B-matrix. For $g\geqslant 3$, the system is overdetermined, but for $g=3$ however, there is a conjecture that the solution exists also for each Riemann B-matrix.

This problem is closely connected with the Novikov hypothesis relative to the Korteweg–de Vries and Kadomtsev–Pietviashvili equations. This hypothesis states that the condition of the solvability of the relevant analogue of (11) (for the KdV or KP equations) is equivalent to the condition which narrows down the class of θ-functions to the so-called θ-functions on the Riemann surfaces. Although this question is recently examined in the theory of abelian functions (e.g. Dubrovin 1981), it will not be discussed further here. Also, for the detailed proof of dispersion equations, the reader is referred to II.

If equation (11) is fulfilled, then formula (1) yields a g-phase quasi-periodic solution of the SG equation in N-dimensional space–time. There is no evidence, however, that it is the unique solution. Equation (11) constitutes also a practical algorithm for the determination of the B-matrix and the scalar products A_{pq}, although the form of coefficients (9) is not convenient for further analysis. Therefore, below, we shall derive a more suitable form of equations (11).

Let us consider the equations

$$\sum_{p,q=1}^{g} A_{pq}\Omega_{\varepsilon}^{pq}=\tfrac{1}{4}(c^{\pm}\delta_{\varepsilon,0}\mp\delta_{\varepsilon,d}), \tag{12}$$

for each $\varepsilon\in D^g$. Equations (12) and (11) are equivalent, but here we include the case $\varepsilon=0$ (introducing also the additional 'dummy' unknown $c^{\pm}/4$).

Multiplying (12) by $(-1)^{\varepsilon\cdot\mu}$, where $\mu\in D^g$ and summing over the elements of the die $\varepsilon\in D^g$, we obtain

$$\sum_{p,q=1}^{g} f_{\mu,pq}A_{pq}-\tfrac{1}{2}[c^{\pm}\mp(-1)^{\mu\cdot d}]f_{\mu}=0, \tag{13}$$

where

$$f_{\mu}=\theta(B\mu|2B), \tag{14}$$

$$f_{\mu,pq}=\partial_{w_p}\partial_{w_q}[\theta(2w+B\mu|2B)\exp(2\pi i w\cdot\mu)]_{w=0}. \tag{15}$$

Observe that the coefficients f_{μ} and $f_{\mu,pq}$ are determined by the values of the θ-function and its derivatives at fixed points. By the relations (A1.5) and next (A1.2) equation (13) becomes now

$$\sum_{p,q=1}^{g}\sum_{\delta\in D^g}(-1)^{\delta\cdot\mu}[\partial_{w_p}\partial_{w_q}\theta(w+\tfrac{1}{2}\delta|B/2)]|_{w=0}A_{pq}$$
$$-\tfrac{1}{2}[c^{\pm}\mp(-1)^{\mu\cdot d}]\sum_{\delta\in D^g}(-1)^{\mu\cdot\delta}\theta(\delta/2|B/2)=0. \tag{16}$$

Once more multiplying by $(-1)^{\varepsilon\cdot\mu}$ and summing over $\mu\in D^g$, the dispersion equation reads finally

$$\sum_{p,q=1}^{g}\theta_{pq}(\varepsilon/2|B/2)A_{pq}=\tfrac{1}{2}[c^{\pm}\theta(\varepsilon/2|B/2)\mp\theta((d-\varepsilon)/2|B/2)], \tag{17}$$

and ought to be satisfied for each $\varepsilon\in D^g$.

$\theta_{pq}(\varepsilon/2|B/2)$ denotes the second (mixed) derivative of the θ-function with respect to arguments w_p and w_q at points $w=\varepsilon/2$.

Relation (17) constitutes a system of 2^g algebraic equations linear with respect to the $[g(g+1)/2]+1$ quantities A_{pq} and $c^{\pm}$, which one can treat as unknown. Observe, however, that the matrix B must also be determined. For $g=1$ or 2, as in the case of equations (11), the system (17) always has a solution for any Riemann matrix B. For $g>2$, the number of equations 2^g exceeds the number $[g(g+1)/2]+1$ of quantities A_{pq} and $c^{\pm}$, and treating the matrix B as a parameter, the system is overdetermined. But the numerical experiment performed by Hirota and Ito (1981) for the KdV equation ($g=3$), where a similar situation occurs, exhibits that some equations are superfluous, and then one hopes that our equation (17) has also a non-trivial solution. This would indicate that the Novikov hypothesis previously mentioned may also be valid for the SG equation.

Comparing (11) and (17), it is seen that the latter relation is much better for numerical evaluations.

If equations (17) have a solution, choosing a_{pj} which satisfy (4), we obtain the completely determined solution of the SG equation in the form (1). In general it is complex and the problem of an extraction of the real solutions seems to be hard.

The solution (1) depends on B doubly: directly, since the matrix B appears in the definition of the θ-function as a parameter (see (A1.1)), and indirectly, since a_{pj} in (2) depends on B through A_{pq}, as the solutions of the dispersion equations (11) or (17). In consequence, it is hard to predict the condition for the reality of solutions. One can ask however, when Ψ defined by (1) is real.

It is seen from the definition (A1.1) of the θ-function that putting

$$B_{ij} = B'_{ij} + \mathrm{i}B''_{ij}, \tag{18}$$

Ψ is real if

$$\begin{aligned} &\text{(i)} \quad \boldsymbol{z} = \boldsymbol{z}' \\ &\text{(ii)} \quad \boldsymbol{z} = \mathrm{i}\boldsymbol{z}'' \end{aligned} \Bigg\} \quad \text{and} \quad B'_{ij} = \begin{cases} \frac{1}{2} & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \tag{19}$$

$$\text{(iii)} \quad \boldsymbol{z} = \boldsymbol{d}/4 + \mathrm{i}\boldsymbol{z}'' \quad \text{and} \quad B'_{ij} = 0 \tag{20}$$

where all the quantities z', z'', B'_{ij}, B''_{ij} are real.

These conditions immediately follow from the requirement $\theta(\boldsymbol{d}/2 + \boldsymbol{z}|B) = \theta^*(\boldsymbol{z}|B)$, where an asterisk denotes the complex conjugate quantity.

In cases (i) and (ii) we have an oscillatory type solution (Ψ is periodical in z_i), while case (iii) describes a rotational type solution (Ψ is monotonic in z''_i). This distinction is originated by one-dimensional solutions describing the oscillations or rotational motion of a pendulum. This one-dimensional case reveals however that (19) and (20) are not the unique requirements to obtain real solutions. Taking into account the unimodular transformation for the Riemann ϑ-function (Bateman and Erdelyi 1955), real solutions can exist also if B' and B'', which are then numbers, obey the equation

$$(B' - b_0)^2 + (B'')^2 = r^2, \tag{21}$$

with suitably chosen b_0 and r. Equation (21) represents a family of circles and the conditions (19), (20) in the one-dimensional case are seen to represent degenerate circles: straight lines.

Similar, although more complex relations occur also in the multidimensional case, that suggest that conditions (19) and (20) are not unique.

3. Relations between the two classes of solutions

Let us consider now the two-phase quasi-periodic solution only. At first the solutions of this type were found much earlier by the substitution

$$\Psi = 4 \tan^{-1} f(u)g(v), \tag{22}$$

where u and v were identified as x and t or as their Lorentz transformations (Seeger *et al* 1953, Perring and Skyrme 1962, Osborne and Stuart 1978, 1980, Zagrodziński 1976).

As is known, (22) leads to first-order differential equations for f and g, and periodical solutions of these equations are expressed by means of elliptic functions. Properties of the solution (22) in application to the nonlinear phenomena in optics have been discussed by many authors (e.g. Lamb 1971, 1980) and in application to the theory of the Josephson junction were analysed also by Parmentier (1978), Costabile *et al* (1978) etc.

Quasi-periodic solutions in the form (22) are divided into three fundamental groups and are known as plasma, breather and fluxon oscillations.

A natural question now occurs: what is the relation between the solutions (1) and (22), if any?

In I and II we made a guess that the two forms are equivalent only if the two-dimensional θ-function has a representation in terms of one-dimensional ϑ-functions, since the periodic solution (22) can always be expressed by the latter.

There are a few situations when the two-dimensional θ-function has such a representation, i.e. can be expressed by a finite sum of the elements containing one-dimensional Riemann ϑ-functions (ϑ_3, ϑ_4). Some of them are listed in appendix 2, but the most important for our purposes is the condition

$$B_{11} = B_{22}. \tag{23}$$

As is shown in appendix 2, if (23) holds, then the two-dimensional θ-function has the representation

$$\theta(z|B) = 2^{-1} \sum_{\delta=0,1} \vartheta_3(z^+ + \tfrac{1}{2}\delta|b^+)\vartheta_3(z^- + \tfrac{1}{2}\delta|b^-), \tag{24}$$

where

$$z^{\pm} = (z_1 \pm z_2)/2, \tag{25}$$

$$b^{\pm} = (B_{11} \pm B_{12})/2 = (B_{22} \pm B_{12})/2, \tag{26}$$

if $\operatorname{Im} b^{\pm} > 0$. $\vartheta_3(z|b)$ denotes the one-dimensional Riemann theta-function.

The equivalence of (1) and (22) requires that

$$f(u)g(v) = \mathrm{i}(\theta(z + \tfrac{1}{2}\boldsymbol{d}|B) - \theta(z|B))/(\theta(z + \tfrac{1}{2}\boldsymbol{d}|B) + \theta(z|B)), \tag{27}$$

which means that the right-hand side of (27) must be 'factorisable' with arguments u, v (of the factors) being linearly independent combinations of z_1 and z_2.

Among the conditions quoted in appendix 2, only (23) fulfils these requirements and by (24), the condition (27) reduces to

$$\mathrm{i}h^+h^- = gf, \tag{28}$$

where

$$h^{\pm} = h^{\pm}(z^{\pm}|b^{\pm}) = (\vartheta_4(z^{\pm}|b^{\pm}) - \vartheta_3(z^{\pm}|b^{\pm}))/(\vartheta_4(z^{\pm}|b^{\pm}) + \vartheta_3(z^{\pm}b^{\pm})). \tag{29}$$

We have used here the commonly accepted notation for the Jacobi theta-functions, writing ϑ_3 and ϑ_4 instead of $\theta(z|b)$ and $\theta(z + \frac{1}{2}|b)$, respectively (Jahnke *et al* 1960).

Each of the functions $h^{\pm}$ can be written as

$$h = [(k')^{1/2} - \mathrm{dn}(u, k)][(k')^{1/2} + \mathrm{dn}(u, k)]^{-1}, \tag{30}$$

where

$$k' = \vartheta_4^2(0|b)/\vartheta_3^2(0|b), \tag{31}$$

$$k = [1 - (k')^2]^{1/2}, \tag{32}$$

$$u = \pi\vartheta_3^2(0|b)z, \tag{33}$$

i.e. is expressible by the elliptic functions.

Making use of the elementary transformation of elliptic functions, (30) can be reduced to the usual form of the periodical solution of the SG equation. For example,

$$h = -A_h \operatorname{cn}(u_h, k_h), \tag{34}$$

where

$$A_h = [1-(k')^{1/2}][1+(k')^{1/2}]^{-1}, \tag{35}$$

$$k_h = \mathrm{i}2^{-3/2}[1-(k')^{1/2}]^2(k')^{-1/4}(1+k')^{-1/2}, \tag{36}$$

$$u_h = 2^{1/2}(1+k')^{1/2}(k')^{1/4}u. \tag{37}$$

Thus we have shown that if the diagonal elements of the matrix B are equal, the two-phase quasi-periodic solutions of the SG equation can be transformed into the form (22). If the diagonal elements are different, however, relation (1) represents a new class of solutions (than expressible by (22)) and we intend to present its properties separately. Here we mention only that, for example, taking the solutions which in the moving frame would be essentially quasi-periodical (with $B_{11} = B_{22}$), by a suitable choice of the diagonal elements of the B matrix ($B_{11} \neq B_{22}$), they can be transformed into strictly periodic solutions without change of their magnitudes (Jaworski and Zagrodziński 1982).

4. Conclusions

In § 2 we have presented a new and simpler version of the system of algebraical dispersion equations, whose solution determines the solution of the multidimensional SG equation.

Another interesting point here is that in the simplest, two-periodical case, the solutions in the form (1), expressed in terms of the abstract theta-functions, constitute a broader class than the solutions used so far in the descriptions of physical phenomena (equation (22)). The conditions for the parameter matrix B when both categories of solutions are equivalent are also reported.

The results derived here can also be achieved by applying the more general class of theta-functions, the so-called theta-functions with characteristics, similarly as was done for the equations of KdV type by Dubrovin (1981).

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Appendix 1

In order to make our paper more accessible to the reader we report the fundamental algebraical properties (only) of θ-functions. More details, particularly relative to the theory of multidimensional abelian functions, can be found in Matveev (1976), Dubrovin (1981), Zakharov *et al* (1980), or in the other papers cited there.

For the algebraical properties of θ-functions, the reader is referred to Krazer (1903), although some of the relations are for the first time quoted in I and II.

The abstract θ-function is defined for complex g-dimensional vectors $\boldsymbol{z}$ by

$$\theta(\boldsymbol{z}|B)=\sum_{\boldsymbol{n}\in Z^g}\exp(\mathrm{i}2\pi\boldsymbol{z}\cdot\boldsymbol{n}+\mathrm{i}\pi\boldsymbol{n}\cdot B\boldsymbol{n}), \tag{A1.1}$$

where the sum over $\boldsymbol{n}\in Z^g$ is taken over all g-dimensional vectors $\boldsymbol{n}$ with the integer components n_i running $-\infty<n_i<\infty$. The matrix B, which plays the role of parameter, is the Riemann matrix, i.e. it is complex and symmetrical with positively defined imaginary part: $\mathrm{Im}\,\boldsymbol{n}\cdot B\boldsymbol{n}>0$. This requirement ensures the absolute convergence of the series (A1.1) in compact domains and analyticity of $\theta(\boldsymbol{z}|B)$.

As previously, by D^g we denote the set of all g-dimensional vectors whose components take only two values: 0 or 1. Then the following identities hold:

$$\theta(\boldsymbol{z}+\boldsymbol{n}+B\boldsymbol{m}|B)=\exp[-\mathrm{i}\pi(2\boldsymbol{z}\cdot\boldsymbol{m}+\boldsymbol{m}\cdot B\boldsymbol{m})]\theta(\boldsymbol{z}|B)\qquad \text{for } \boldsymbol{m},\boldsymbol{n}\in Z^g, \tag{A1.2}$$

$$\theta^2(\boldsymbol{z}|B)=\sum_{\boldsymbol{\varepsilon}\in D^g}\exp[\mathrm{i}\pi(2\boldsymbol{\varepsilon}\cdot\boldsymbol{z}+\boldsymbol{\varepsilon}\cdot B\boldsymbol{\varepsilon})]\theta(B\boldsymbol{\varepsilon}|2B)\theta(2\boldsymbol{z}+B\boldsymbol{\varepsilon}|2B), \tag{A1.3}$$

$$\theta(2\boldsymbol{z}|2B)=2^{-g}\theta^{-1}(0|2B)\sum_{\boldsymbol{\varepsilon}\in D^g}\theta^2(\boldsymbol{z}+\tfrac{1}{2}\boldsymbol{\varepsilon}|B), \tag{A1.4}$$

$$\theta(2\boldsymbol{z}|4B)=2^{-g}\sum_{\boldsymbol{\varepsilon}\in D^g}\theta(\boldsymbol{z}+\tfrac{1}{2}\boldsymbol{\varepsilon}|B), \tag{A1.5}$$

$$\theta(\boldsymbol{z}_2-\boldsymbol{z}_1|2B)\theta(\boldsymbol{z}_2+\boldsymbol{z}_1|2B)=2^{-g}\sum_{\boldsymbol{\varepsilon}\in D^g}\theta(\boldsymbol{z}_1+\tfrac{1}{2}\boldsymbol{\varepsilon}|B)\theta(\boldsymbol{z}_2+\tfrac{1}{2}\boldsymbol{\varepsilon}|B), \tag{A1.6}$$

$$\theta(\boldsymbol{z}|B)=(-\mathrm{i})^{-g/2}(\det B)^{-1/2}\exp(-\mathrm{i}\pi\boldsymbol{z}\cdot B^{-1}\boldsymbol{z})\theta(B^{-1}\boldsymbol{z}|-B^{-1}), \tag{A1.7}$$

$$\partial_{z_p}\ln\frac{\theta(\boldsymbol{z}+\frac{1}{4}\boldsymbol{\delta}|B)}{\theta(\boldsymbol{z}-\frac{1}{4}\boldsymbol{\delta}|B)}=\sum_{\boldsymbol{\varepsilon}\in D^g}\Omega^p_{\boldsymbol{\varepsilon}}(\boldsymbol{\delta})\frac{\theta^2(\boldsymbol{z}+\frac{1}{2}\boldsymbol{\varepsilon}|B)}{\theta(\boldsymbol{z}+\frac{1}{4}\boldsymbol{\delta}|B)\theta(\boldsymbol{z}-\frac{1}{4}\boldsymbol{\delta}|B)}, \tag{A1.8}$$

$$\partial_{z_p}\partial_{z_q}\ln\theta(\boldsymbol{z}|B)=\sum_{\boldsymbol{\varepsilon}\in D^g}\Omega^{pq}_{\boldsymbol{\varepsilon}}\frac{\theta^2(\boldsymbol{z}+\frac{1}{2}\boldsymbol{\varepsilon}|B)}{\theta^2(\boldsymbol{z}|B)}, \tag{A1.9}$$

where the constants $\Omega^p_{\boldsymbol{\varepsilon}}$ and $\Omega^{pq}_{\boldsymbol{\varepsilon}}$ are given by

$$\Omega^p_{\boldsymbol{\varepsilon}}=2^{-g}\sum_{\boldsymbol{\mu}\in D^g}(-1)^{\boldsymbol{\varepsilon}\cdot\boldsymbol{\mu}}\theta^{-1}(B\boldsymbol{\mu}|2B)\partial_{w_p}[\theta(2\boldsymbol{w}+B\boldsymbol{\mu}|2B)\exp(\mathrm{i}2\pi\boldsymbol{w}\cdot\boldsymbol{\mu})]|_{\boldsymbol{w}=-\boldsymbol{\delta}/4}, \tag{A1.10}$$

$$\Omega^{pq}_{\boldsymbol{\varepsilon}}=2^{-(g+1)}\sum_{\boldsymbol{\mu}\in D^g}(-1)^{\boldsymbol{\varepsilon}\cdot\boldsymbol{\mu}}\theta^{-1}(B\boldsymbol{\mu}|2B)\partial_{w_p}\partial_{w_q}[\theta(2\boldsymbol{w}+B\boldsymbol{\mu}|2B)\exp(\mathrm{i}2\pi\boldsymbol{w}\cdot\boldsymbol{\mu})]|_{\boldsymbol{w}=0}, \tag{A1.11}$$

respectively. (A1.10) and (A1.11) mean that the coefficients $\Omega^p_{\boldsymbol{\varepsilon}}$, $\Omega^{pq}_{\boldsymbol{\varepsilon}}$ are determined by the derivatives of the θ-function, but at fixed points.

Appendix 2

If the matrix B, being the Riemann matrix, takes the particular forms, the two-dimensional θ-function can always be expressed by a finite sum involving one-dimensional θ-functions, i.e. by commonly used in the theory of elliptic functions, the Reimann theta-functions ϑ_3 called then the Jacobi theta-functions.

From the definition (A1.1) and by the relation

$$\vartheta_3(qz|q^2b) = q^{-1}\sum_{s=0}^{q-1} \vartheta_3\left(z + \frac{s}{q}\middle|b\right), \tag{A2.1}$$

where q is integer, it is easy to check that if

$$\text{(a)} \qquad B_{12} = p/q, \tag{A2.2}$$

where p and q are integers, then, for $z = (z_1, z_2)$,

$$\theta(z|B) = q^{-1}\sum_{s=0}^{q-1}\sum_{t=0}^{q-1} \exp\left(-i2\pi\frac{st}{q}\right) \vartheta_3\left(z_1 + \frac{s}{q}\middle|B_{11}\right) \vartheta_3\left(z_2 + \frac{pt}{q}\middle|B_{22}\right). \tag{A2.3}$$

If

$$\text{(b)} \qquad B_{12} = B_{22}r/q \qquad (\text{or } B_{12} = B_{11}r/q), \tag{A2.4}$$

and r, q are integers, then

$$\theta(z|B) = \sum_{s=0}^{q-1} \exp[i\pi(2sz_1 + s^2B_{11})]\vartheta_3\left(qz_1 - rz_2 + \frac{s}{q}a\middle|a\right) \vartheta_3\left(z_2 + \frac{s}{q}rB_{22}\middle|B_{22}\right), \tag{A2.5}$$

where $a = q^2B_{11} - r^2B_{22}$ and $\mathrm{Im}\, a > 0$ (or relation (A2.5) with exchanged indices $1 \rightleftarrows 2$). Moreover, the requirements (a) and (b) can be written jointly as

$$\text{(c)} \qquad B_{12} = p + rB_{22}/q \qquad (\text{or } B_{12} = p + rB_{11}/q), \tag{A2.6}$$

where p, q, r are integers.

Similar relations can be derived upon application of (a) or (b) to the inverse matrix $(-B^{-1})$ due to the equation (A1.7). Thus the other case of interest, when a two-dimensional θ-function can be 'split' into one-dimensional θ-functions, is

$$\text{(d)} \qquad B_{12} = (B_{11}B_{22} - B_{12}^2)p/q + B_{11}r/q, \tag{A2.7}$$

with p, q, r integers (or a similar relation with interchanged indices).

The most interesting however, is the case

$$\text{(e)} \qquad B_{11} = B_{22}. \tag{A2.8}$$

Assuming that $\mathrm{Im}(B_{11} \pm B_{12}) > 0$, by the definition (A1.1) we have

$$\begin{aligned}\theta(z|B) &= \sum_{m,n\in Z} \exp\{i\pi[2(mz_1 + nz_2) + (m^2 + n^2)B_{11} + 2mnB_{12}]\} \\ &= \sum_{p\in Z} \exp[i\pi(2pz_2 + p^2B_{11})]\vartheta_3(z_1 - z_2 - p(B_{11} - B_{12})|2(B_{11} - B_{12})),\end{aligned} \tag{A2.9}$$

where in the first part of relation (A2.9), $p = m + n$ was substituted. Making use of (A1.5) and next (A1.2), (A2.9) yields

$$\begin{aligned}&2^{-1}\sum_{p\in Z} \exp[i\pi(pz_2 + p^2B_{11})] \sum_{\delta=0}^{1} \vartheta_3([z_1 - z_2 - p(B_{11} - B_{12}) + \delta]/2|(B_{11} - B_{22})/2) \\ &\quad = 2^{-1}\sum_{\delta=0}^{1} \vartheta_3(z^+ + \delta/2|b^+)\vartheta_3(z^- + \delta/2|b^-),\end{aligned} \tag{A2.10}$$

where $z^{\pm}$ and $b^{\pm}$ are given by (25) and (26), respectively.

This concludes the proof of the identity (24).

References

Bateman H and Erdelyi 1955 *Higher Transcendental Functions* (New York: McGraw Hill)
Costabile G, Parmentier R D, Savo B, McLaughlin D W and Scott A C 1978 *Appl. Phys. Lett.* **32** 587
Dubrovin B A 1981 *Usp. Mat. Nauk* **36** 11
Forest G and McLaughlin 1980 *Spectral Theory for Periodic SGE, Preprint* Los Alamos Scientific Laboratory
Hirota R and Ito M 1981 *J. Phys. Soc. Japan* **50** 338
Jahnke E, Emde F and Losch F 1960 *Tafeln Hoherer Funktionen* (Stuttgart: Teubner)
Jaworski M and Zagrodziński J 1982 to appear
Kozel V O and Kotlyarov V P 1976 *Dokl. Akad. Nauk. Ukr. SSR* ser. A **10** 878
Krazer A 1903 *Lehrbuch der Thetafunktionen* (Leipzig: Teubner)
Lamb G 1971 *Rev. Mod. Phys.* **43** 99
—— 1980 *Elements of Soliton Theory* (New York: Wiley)
Matveev V B 1976 *Abelian Functions and Solitons, Preprint* 373 of Institute of Theoretical Physics, University of Wrocław
Nakamura A 1979 *J. Phys. Soc. Japan* **47** 1701
—— 1980 *J. Phys. Soc. Japan* **48** 1365
Osborne A and Stuart A E 1978 *J. Math. Phys.* **19** 1573
—— 1980 *J. Math. Phys.* **21** 726
Parmentier R D 1978 *Fluxons in Long Josephson Junction* in *Soliton in Action* ed K Lonngren and A Scott (New York: Academic)
Perring J K and Skyrme T H 1962 *Nucl. Phys.* **31** 550
Seeger A, Donth H and Kochendörfer A 1953 *Z. Phys.* **134** 173
Zagrodziński J 1976 *Phys. Lett.* **57A** 213
—— 1981 *Lett. Nuovo Cimento* **30** 266
—— 1981 *A method of direct substitution for periodic solutions of multidimensional* SGE, *Workshop on NL Evolution Equations, Solitons and Spectral Methods, Trieste* (to be published in *Physica* D)
—— 1982 *J. Math. Phys.* **23** to appear
Zakharov V E, Manakov S V, Novikov S P and Pitajevskij L P 1980 *Theory of Solitons* (Moskva: Nauka) (in Russian)